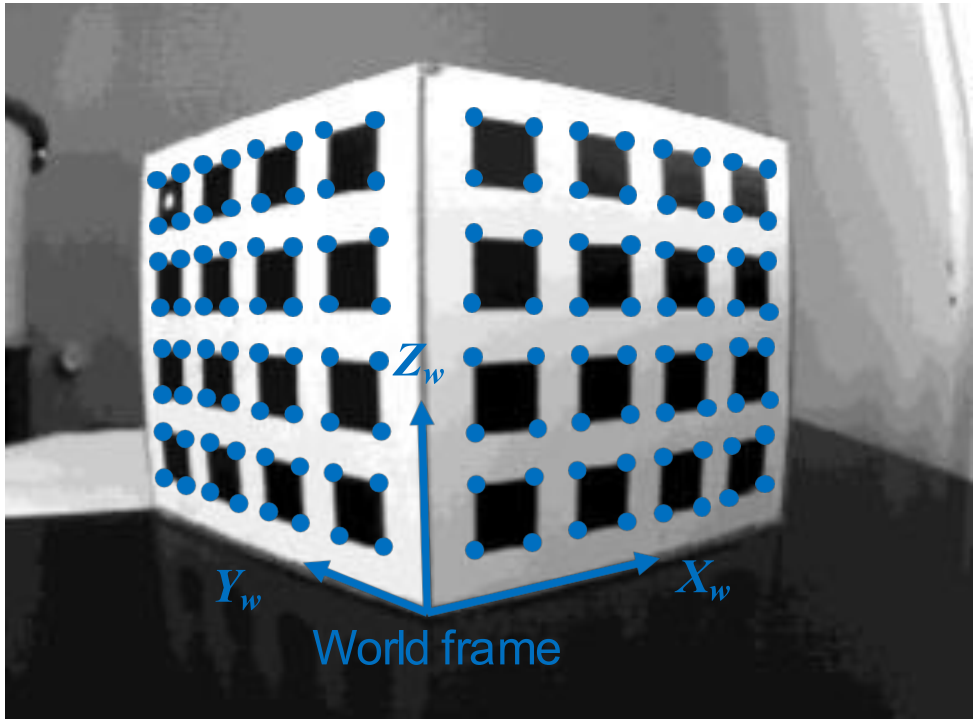
Tsai's Method: Calibration from 3D Objects

- This method was proposed in 1987 by Tsai and consists of measuring the 3D position of $n \geq 6$ **control points** on a 3D calibration target and the **2D coordinates of their projection** in the image.



Tsai, Roger Y. (1987) "A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-Shelf TV Cameras and Lenses," *IEEE Journal of Robotics and Automation*, 1987. PDF.

Applying the Direct Linear Transform (DLT) algorithm

The idea of the DLT is to rewrite the perspective projection equation as a **homogeneous linear equation** and solve it by standard methods. Let's write the perspective equation for a generic 3D-2D point correspondence:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K\left[R \mid T\right] \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u r_{11} + u_0 r_{31} & \alpha_u r_{12} + u_0 r_{32} & \alpha_u r_{13} + u_0 r_{33} & \alpha_u t_1 + u_0 t_3 \\ \alpha_v r_{21} + v_0 r_{31} & \alpha_v r_{22} + v_0 r_{32} & \alpha_v r_{23} + v_0 r_{33} & \alpha_v t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

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$$\Rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

What are the assumptions behind this this substitution?

Applying the Direct Linear Transform (DLT) algorithm

$$\underbrace{\begin{pmatrix} X_w^1 & Y_w^1 & Z_w^1 & 1 & 0 & 0 & 0 & 0 & -u_1 X_w^1 & -u_1 Y_w^1 & -u_1 Z_w^1 & -u_1 \\ 0 & 0 & 0 & 0 & X_w^1 & Y_w^1 & Z_w^1 & 1 & -v_1 X_w^1 & -v_1 Y_w^1 & -v_1 Z_w^1 & -v_1 \\ & & & & & & & \vdots & & & & \\ X_w^n & Y_w^n & Z_w^n & 1 & 0 & 0 & 0 & 0 & -u_n X_w^n & -u_n Y_w^n & -u_n Z_w^n & -u_n \\ 0 & 0 & 0 & 0 & X_w^n & Y_w^n & Z_w^n & 1 & -v_n X_w^n & -v_n Y_w^n & -v_n Z_w^n & -v_n \end{pmatrix}}_{\text{Q (this matrix is } \textbf{known}\text{)}} \underbrace{\begin{pmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{pmatrix}}_{\text{M (this matrix is } \textbf{unknown}\text{)}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad \mathrm{Q} \cdot \mathrm{M} = 0$$

Applying the Direct Linear Transform (DLT) algorithm

$$Q \cdot M = 0$$

Minimal solution

- $Q_{(2n\times12)}$ should have rank 11 to have a unique (up to a scale) *non-zero* solution M
- Because each 3D-to-2D point correspondence provides 2 independent equations, then $5+\frac{1}{2}$ point correspondences are needed (in practice **6 point** correspondences!)

Over-determined solution

- For $n \geq 6$ points, a solution is the **Least Square solution**, which minimizes the sum of squared residuals, $||QM||^2$, subject to the constraint $||M||^2 = 1$. It can be solved through Singular Value Decomposition (SVD). The solution is the eigenvector corresponding to the smallest eigenvalue of the matrix Q^TQ (because it is the unit vector x that minimizes $||Qx||^2 = x^TQ^TQx$.
- Matlab instructions:
  - `[U,S,V] = SVD(Q);`
  - `M = V(:,12);`

Applying the Direct Linear Transform (DLT) algorithm

- Once we have determined M, we can recover the intrinsic and extrinsic parameters by remembering that:

$$\mathrm{M} = \mathrm{K}(\mathrm{R} \mid \mathrm{T})$$

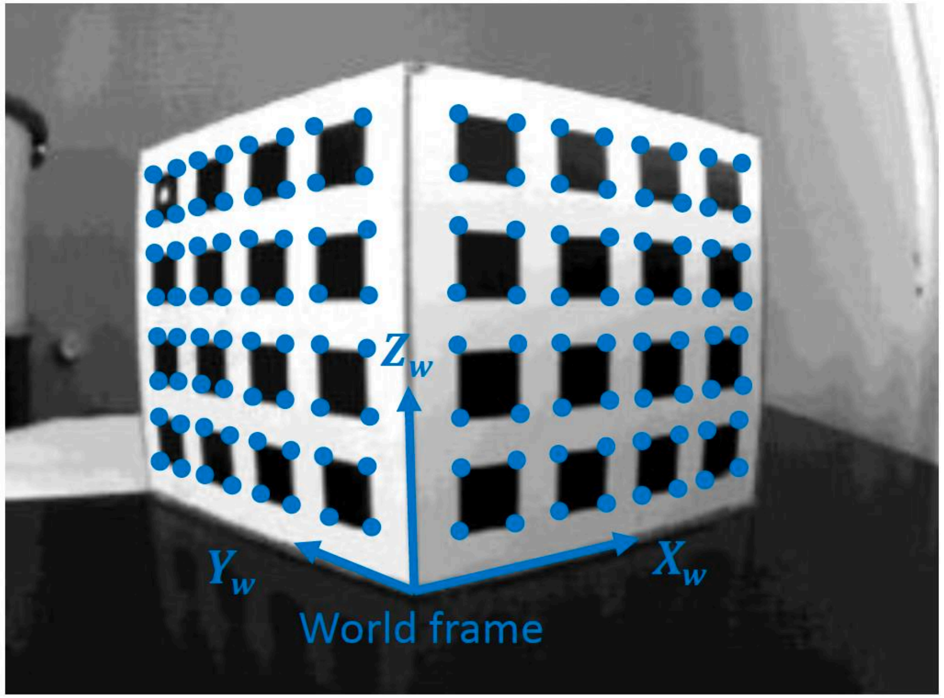
$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

Considering the first three columns of M, it is equal to K R,
the product of an upper triangular matrix and an orthogonal matrix

We can use the QR decomposition from linear algebra

Example of Tsai's Calibration Results

Recommendation: use many **more than 6 points** (ideally more than 20) **and non coplanar**



Corners can be detected with accuracy < 0.1 pixels (see Lecture 5)

α_u	α_u/α_v	K_{12}	u_0	v_0	**Average Reprojection error**
1673.3	1.0063	1.39	379.96	305.78	0.365

Why is this ratio not 1?

What is this?

What is this?

How can we estimate the lens distortion parameters?

How can we enforce $\alpha_u = \alpha_v$ and $K_{12} = 0$?

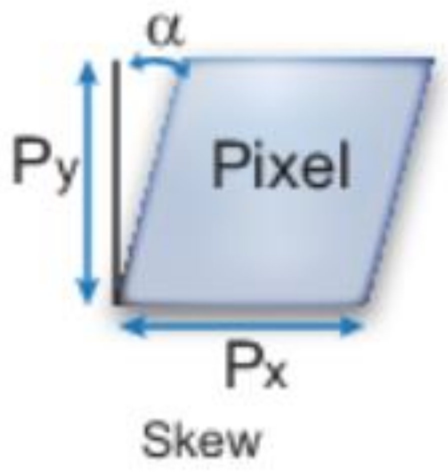
Intrinsic Parameters

(figures from https://www.mathworks.com/help/vision/ug/camera-calibration.html)

$$\begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

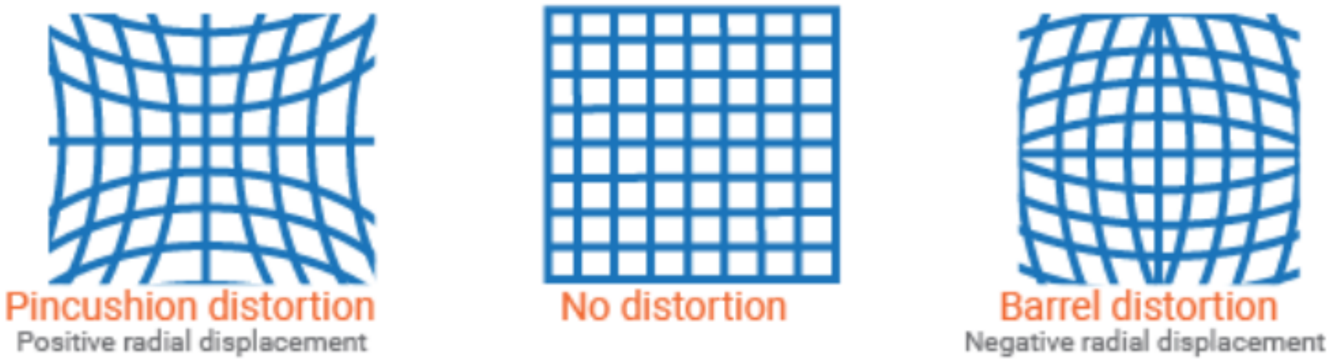
$[c_x \quad c_y]$ – Optical center (the principal point), in pixels.
$(f_x, \quad f_y)$ – Focal length in pixels. $f_x = F/p_x$ $f_y = F/p_y$ F – Focal length in world units, typically expressed in millimeters. $(p_x, \quad p_y)$ – Size of the pixel in world units.
s – Skew coefficient, which is non-zero if the image axes are not perpendicular. $s = f_x \tan \alpha$

The pixel skew is defined as:



Non-linear Lens Distortion

(figures from https://www.mathworks.com/help/vision/ug/camera-calibration.html)



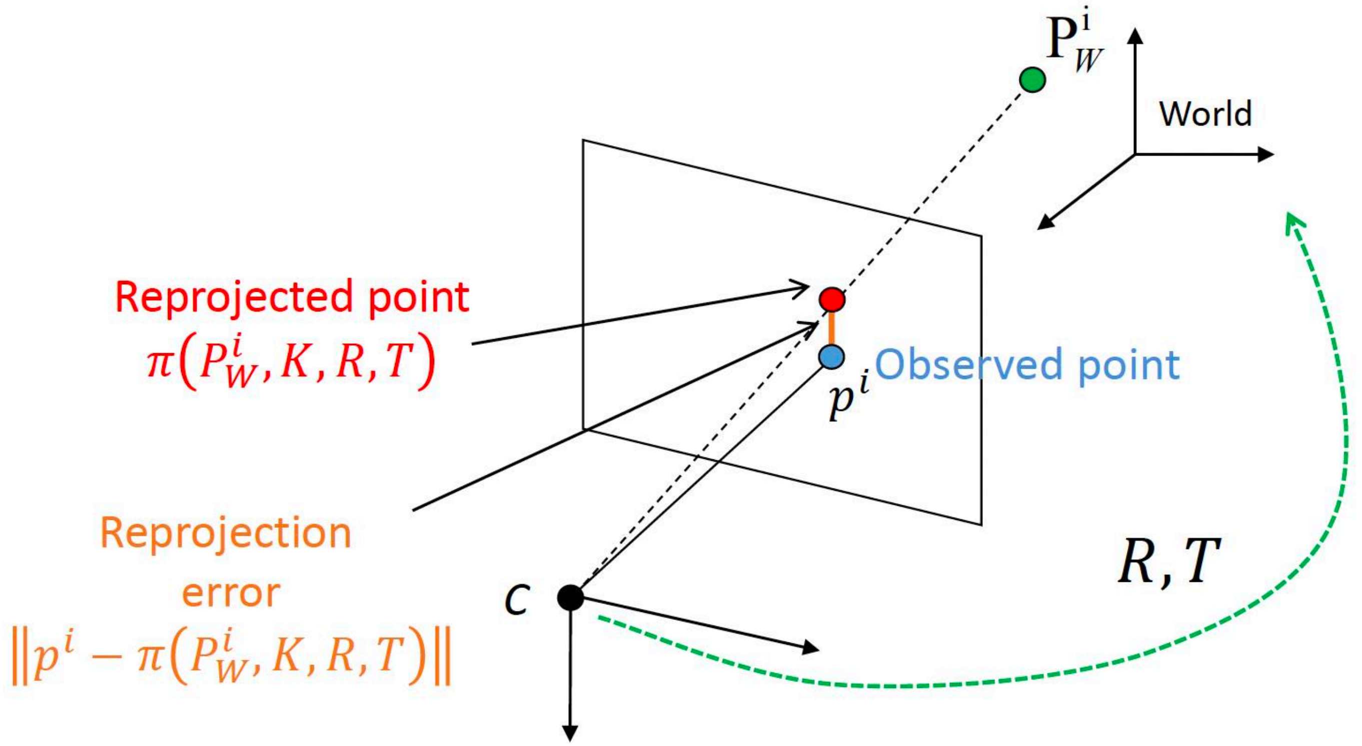
The radial distortion coefficients model this type of distortion. The distorted points are denoted as ($x_{\text{distorted}}$, $y_{\text{distorted}}$):

$$x_{\text{distorted}} = x(1 + k_1*r^2 + k_2*r^4 + k_3*r^6)$$

$$y_{\text{distorted}} = y(1 + k_1*r^2 + k_2*r^4 + k_3*r^6)$$

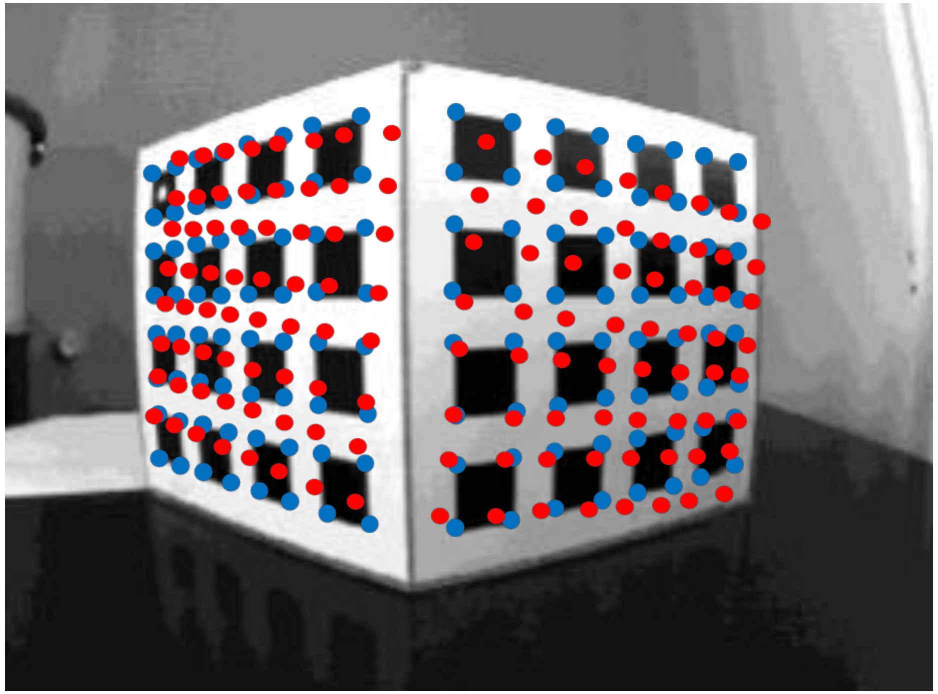
Reprojection Error

- The reprojection error is the **Euclidean distance** (in pixels) between an **observed image point** and the **corresponding** 3D point **reprojected** onto the camera frame.
- The reprojection error gives us a **quantitative measure of the accuracy** of the calibration (**ideally it should be zero**).



Reprojection Error

- The reprojection error can be used to assess the quality of the camera calibration
- What reprojection error is acceptable?
- What are the sources of the reprojection error?
- How can we further improve the calibration parameters?



Control points (observed points)

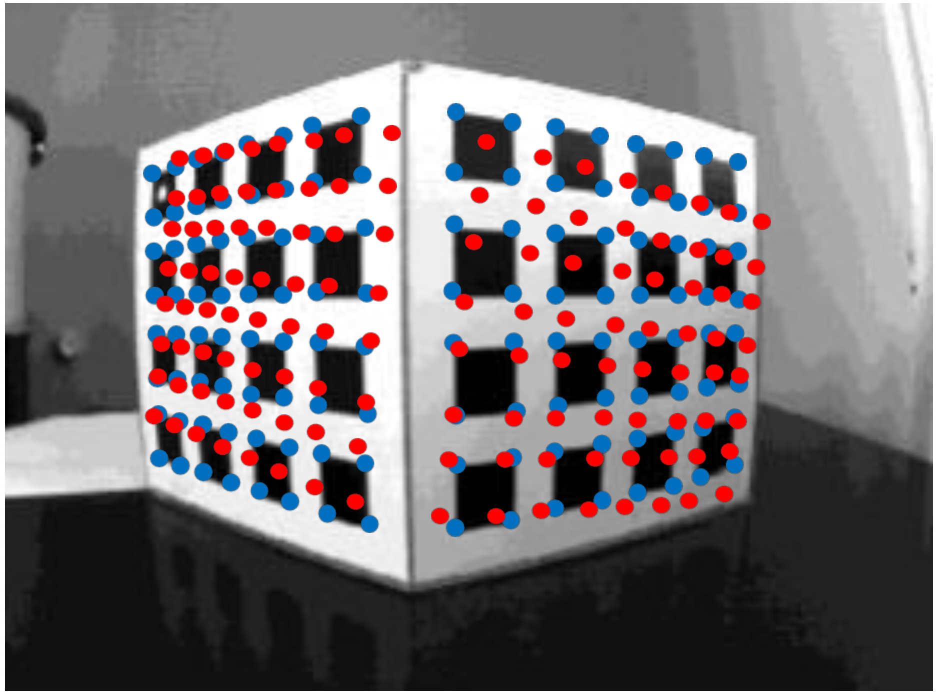
Reprojected points $\pi\left(P_W^i, K, R, T\right)$

Non-Linear Calibration Refinement

- The calibration parameters K, R, T determined by the DLT can be refined by minimizing the following cost:

$$K, R, T, lens\ distortion = argmin_{K,R,T,lens} \sum_{i=1}^{n} \left\| p^i - \pi\left(P_W^i, K, R, T\right) \right\|^2$$

- This time we also include the **lens distortion** (can be set to 0 for initialization)
- Can be minimized using **Levenberg–Marquardt** (more robust than Gauss-Newton to local minima)



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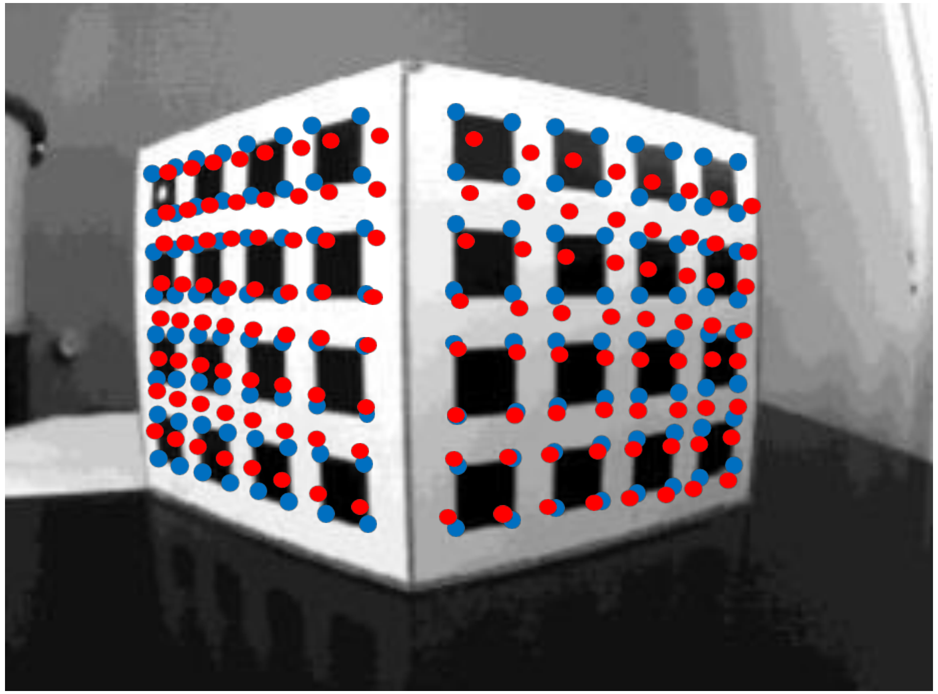

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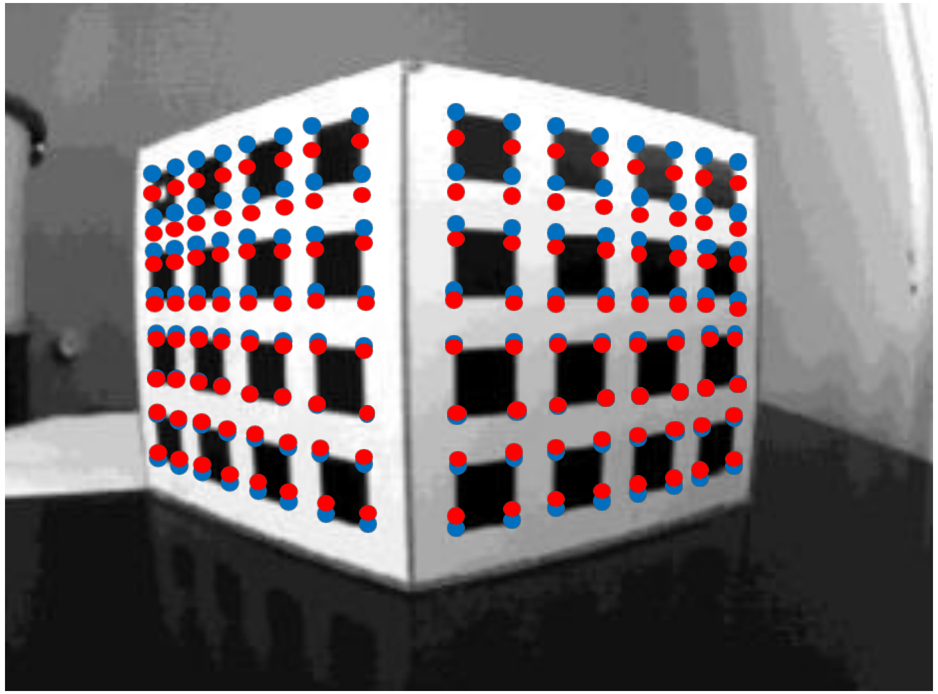

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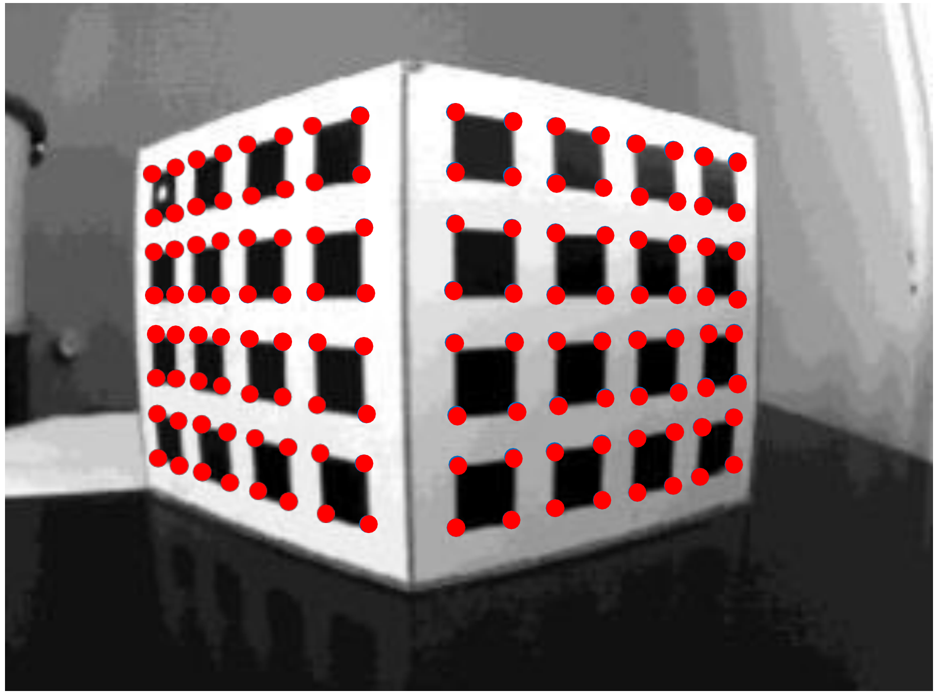

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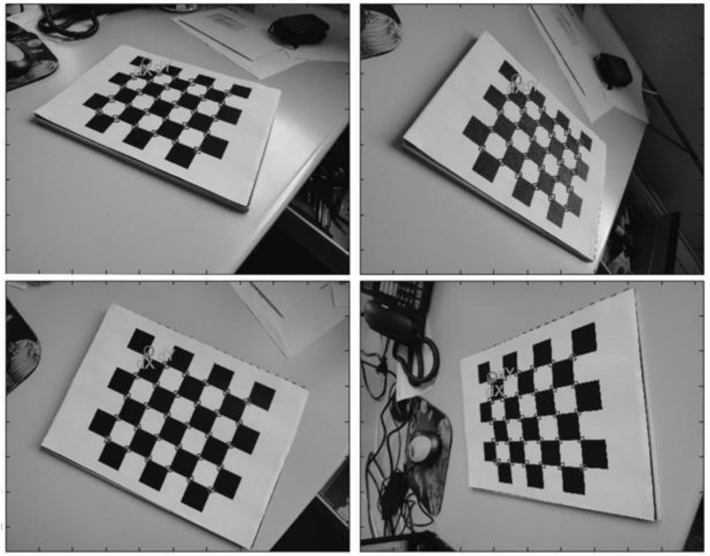
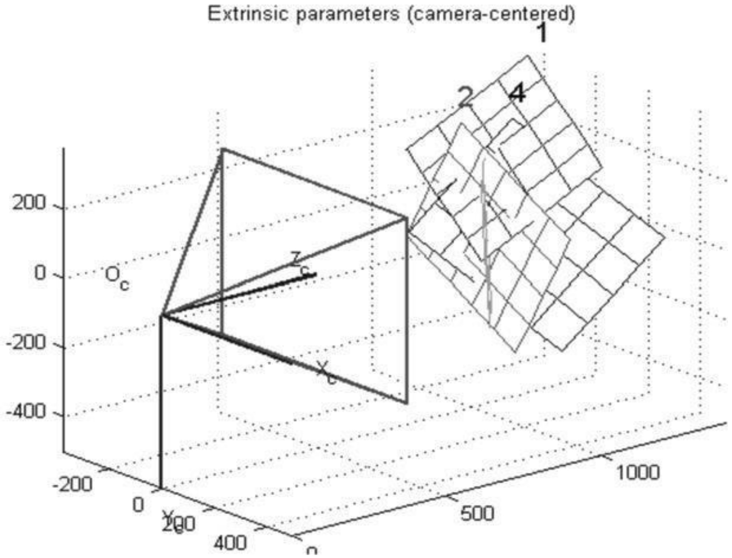
Zhang's Algorithm: Calibration from Planar Grids

- **Tsai's calibration** requires that the world's 3D points are non-coplanar, which is **not very practical**
- **Today's camera calibration toolboxes** (Matlab, OpenCV) use **multiple views** of a **planar grid** (e.g., a checker board)
- They are based on a method developed in 2000 by Zhang (Microsoft Research)





Zhang, A flexible new technique for camera calibration, IEEE Transactions on Pattern Analysis and Machine Intelligence, 2000. PDF.

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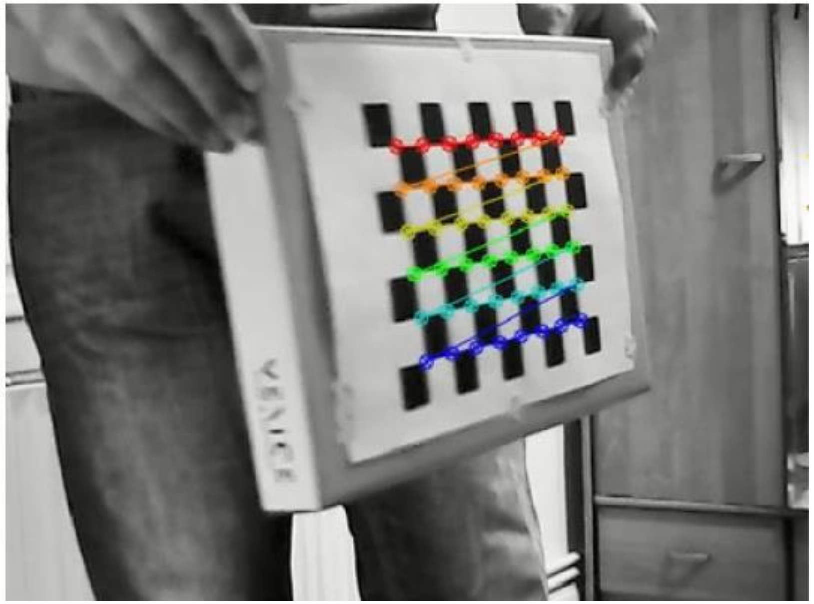

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Applying the Direct Linear Transform (DLT) algorithm

As in Tsai's method, we start by writing the perspective projection equation (again, we neglect the radial distortion). However, in **Zhang's method the points are all coplanar**, i.e., $\boldsymbol{Z_w = 0}$, and thus we can write:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K\left[R \mid T\right] \cdot \begin{bmatrix} X_w \\ Y_w \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 0 \\ 1 \end{bmatrix}$$

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$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = H \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

This matrix is called Homography

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_1^{\mathrm{T}} \\ h_2^{\mathrm{T}} \\ h_3^{\mathrm{T}} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

where h_i^{T} is the i-*th* row of H

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_1^T \\ h_2^T \\ h_3^T \end{bmatrix} \cdot \underbrace{\begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}}_{\rightarrow P}$$

Conversion back from homogeneous coordinates to pixel coordinates leads to:

$$u = \frac{\lambda u}{\lambda} = \frac{h_1^T \cdot P}{h_3^T \cdot P} \qquad \Rightarrow \qquad (h_1^T - u_i h_3^T) \cdot P_i = 0$$

$$v = \frac{\lambda v}{\lambda} = \frac{h_2^T \cdot P}{h_3^T \cdot P} \qquad \qquad (h_2^T - v_i h_3^T) \cdot P_i = 0$$

Applying the Direct Linear Transform (DLT) algorithm

- By re-arranging the terms, we obtain:

$$\begin{aligned}(h_1^{\mathrm{T}} - u_i h_3^{\mathrm{T}}) \cdot P_i = 0 \\ (h_2^{\mathrm{T}} - v_i h_3^{\mathrm{T}}) \cdot P_i = 0\end{aligned} \quad \Rightarrow \quad \begin{aligned} P_i^{\mathrm{T}} \cdot h_1 + 0 \cdot h_2^{\mathrm{T}} - u_i P_i^{\mathrm{T}} \cdot h_3^{\mathrm{T}} = 0 \\ 0 \cdot h_1^{\mathrm{T}} + P_i^{\mathrm{T}} \cdot h_2^{\mathrm{T}} - v_i P_i^{\mathrm{T}} \cdot h_3^{\mathrm{T}} = 0 \end{aligned} \quad \Rightarrow \begin{pmatrix} P_i^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_i^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_i^{\mathrm{T}} & -v_1 P_i^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- For n points (from a **single view**), we can stack all these equations into a big matrix:

$$\underbrace{\begin{pmatrix} P_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_1^{\mathrm{T}} & -v_1 P_1^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ P_n^{\mathrm{T}} & 0^{\mathrm{T}} & -u_n P_n^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_n^{\mathrm{T}} & -v_n P_n^{\mathrm{T}} \end{pmatrix}}_{\text{Q (this matrix is known)}} \underbrace{\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}}_{\text{H (this matrix is unknown)}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathrm{Q} \cdot \mathrm{H} = 0$$

Q (this matrix is **known**) H (this matrix is **unknown**)

Applying the Direct Linear Transform (DLT) algorithm

$$Q \cdot H = 0$$

Minimal solution

- $Q_{(2n\times 9)}$ should have rank 8 to have a unique (up to a scale) non-trivial solution H
- Each point correspondence provides 2 independent equations
- Thus, a minimum of **4 non-collinear points** is required

Over-determined solution

- $n \geq 4$ points
- It can be solved through Singular Value Decomposition (SVD) (same considerations as before)

How to recover K, R, T

- H can be decomposed by recalling that:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}$$

- Differently from Tsai's, the decomposition of H into K, R, T requires **at least two views**
- In practice **the more views the better**, e.g., 20-50 views spanning the entire field of view of the camera for the best calibration results
- Notice that now **each view j has a different homography** H^j (and so a different R^j and T^j). However, ***K* is the same for all views**:

$$\begin{bmatrix} h_{11}^j & h_{12}^j & h_{13}^j \\ h_{21}^j & h_{22}^j & h_{23}^j \\ h_{31}^j & h_{33}^j & h_{33}^j \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11}^j & r_{12}^j & t_1^j \\ r_{21}^j & r_{22}^j & t_2^j \\ r_{31}^j & r_{32}^j & t_3^j \end{bmatrix}$$